J. Fluid Mech. (1973), *vol.* 53, *part* 3, *pp.* 481–493
Printed in Great Britain

Amplification and decay of long nonlinear waves

By S. LEIBOVICH AND J. D. RANDALL
Upson Hall, Cornell University, Ithaca, N.Y. 14850

(Received 31 July 1972)

The interaction of weakly nonlinear waves with slowly varying boundaries is considered. Special emphasis is given to rotating fluids, but the analysis applies with minor modifications to waves in stratified fluids and shallow-water waves. An asymptotic solution of a variant of the Korteweg–de Vries equation with variable coefficients is developed that produces a 'Green's law' for the amplification of waves of finite amplitude. For shallow-water waves in water of variable depth, the result predicts wave growth proportional to the $-\frac{1}{3}$ power of the depth.

1. Introduction

Interactions between long waves in incompressible fluids with slow and weak variations of geometrical constraints are studied in this paper. It has been shown by Benjamin (1967), Benney (1966) and Leibovich (1970) that the Korteweg–de Vries equation governs inviscid long waves in rotating fluids contained in cylindrical tubes, and in stratified fluids between parallel walls. Other examples of continuous fluid flows that are governed by this equation are known, the most famous of which is the propagation of long waves in shallow water with a horizontal bottom, for which it was first introduced by Korteweg & de Vries (1895). The nonlinear shallow-water wave problem with sloping bottom has been the subject of much work (see, for example, Peregrine 1967; Madsen & Mei 1969), but nearly always starting with a system of equations much more complicated than the Korteweg–de Vries equation. Results from these investigations were of necessity primarily numerical and therefore special.

Recently Kakutani (1971) has shown that a modification of the Korteweg–de Vries equation can describe shallow-water wave propagation over gently sloping bottoms.† In this paper, we derive an equation of the same form to describe weakly nonlinear long-wave propagation in rotating streaming fluids contained in circular tubes of variable area. The area changes are assumed to be small and to occur over distances large compared with a wavelength of the disturbance. This paper therefore extends the results of Leibovich (1970). A precisely analogous derivation can be carried out (cf. Randall 1972) for the case of stratified fluids between non-parallel walls, and for other similar situations. The analysis of the

† A paper by Johnson (1972) appeared after this work was submitted for publication. Reference is made there to a forthcoming publication by Johnson in which a single modified Korteweg–de Vries equation is derived to describe this problem. We have not seen the cited paper.

present paper, although carried out in detail in a special context, therefore is expected to apply to a considerably wider class of problems.

An important distinction, not revealed by the earlier work of Benjamin (1967) and Leibovich (1970), between critical and non-critical flows (in the sense of Benjamin 1962) is found to be necessary. In critical flows a linear solution is not possible, and nonlinear effects must be included from the start. This behaviour has parallels in other problems of fluid motion and is discussed in § 3. Although the equations describing both non-critical and critical flows are derived here, only the non-critical case is treated in detail. The critical case, which may need to involve viscous effects as discussed in § 5, is the subject of the adjacent companion paper (Randall & Leibovich 1973).

Wave propagation in both critical and non-critical flows is described by an equation of the form

$$U_T = \lambda U U_X + \beta U_{XXX} + \mu U$$

(with suitably scaled distance X, time T and disturbance U), where the coefficients λ, β and μ are slowly varying functions of X and T. Kakutani's (1971) equation is of this form, as is an equation found by Leibovich & Randall (1971) in a different context. Amplification or decay occurs depending upon whether μ is positive or negative. When μ is positive, and μT large, the amplifying solution displays 'terminal similarity', with the dominant contribution being of solitary-wave form, but with exponentially growing amplitude:

$$U = a\,\mathrm{sech}^2\left\{\left[\frac{\lambda a}{12\beta}\right]^{\frac{1}{2}}\left(X + \tfrac{1}{3}\lambda\int_0^T a\,dT\right)\right\},$$

where

$$a = a_0 \exp\left(\tfrac{4}{3}\textstyle\int \mu\,dT\right).$$

(The same form of solution is obtained by Ott & Sudan (1970) for the damped case with μ small and negative, and T large.) Computer solutions of the initial-value problem for the full equation (with constant λ, β and μ) corroborate the asymptotic result.

The terminal similarity solution evaluated for Kakutani's (1971) equation yields

$$a = a_0(H_0/H)^{\frac{1}{3}}$$

for shallow-water waves, where H is the local depth of the water. This compares with 'Green's law' (Lamb 1932) for infinitesimal waves of extreme length in which the exponent is $\frac{1}{4}$.

2. Formulation

An axially symmetric, concentrated vortex flow of an incompressible fluid is assumed to occur in a tube of slowly varying area. In focusing upon the effect of tube walls, we tacitly assume that the core region of vorticity centred around the z axis occupies a non-negligible volume of the tube. Thus, the tube wall radius (with a typical value b) is a modest multiple of the core radius, and b will be chosen as reference for radial distance (r). Waves with a length scale $l = b/\kappa$, $\kappa \ll 1$, are of interest, and l is chosen as reference for axial distance (z).

Small changes of order δ in tube area are assumed to occur on a much longer scale $L = l/\alpha$, $\alpha \ll 1$.

Velocities are made dimensionless by comparison with the maximum azimuthal velocity V_0 in the undisturbed flow and time with l/V_0. The area-like co-ordinate $y = r^2/b^2$ is more convenient than radius.

In terms of the stream function Ψ (radial velocity $u = -r^{-1}\Psi_z$ and axial velocity $v = r^{-1}\Psi_r$) and 'circulation' Γ (azimuthal velocity $v = r^{-1}\Gamma$) the governing equations for laminar flow of a fluid with constant kinematic viscosity ν may be written as follows:

$$\left.\begin{aligned} D^2\Psi_t + 2\Psi_y D^2\Psi_z + 2y^{-1}\Gamma\Gamma_z - 2y\Psi_z[y^{-1}D^2\Psi]_y &= \tilde{\mu}D^4\Psi, \\ \Gamma_t + 2\Psi_y\Gamma_z - 2\Psi_z\Gamma_y &= \tilde{\mu}D^2\Gamma, \\ D^2 &\equiv 4y\partial^2/\partial y^2 + \kappa^2\partial^2/\partial z^2, \\ \tilde{\mu} &\equiv \nu/V_0 b\kappa. \end{aligned}\right\} \tag{1}$$

The 'support flow' is defined as that portion of the flow that would occur in the absence of waves. It develops with axial distance owing to impressed wall effects over distances comparable with l/α and to viscous effects over distances comparable with $l/\tilde{\mu}$.

In this paper, we take $\alpha \gg \tilde{\mu}$, so that the changes in the support flow are dominated by wall geometry. In effect then, the support flow is treated as inviscid. We shall be interested, however, in waves whose position may not vary greatly from a fixed wall location. Viscosity may, over a long period of time, affect the wave motion. Viscous terms in (1) will therefore be restored when needed to describe wave damping.

Let $x = \alpha z$, and let the tube wall be located at $Y = 1 + \delta h(x)$, where h is a prescribed function of bounded variation. It is known (Benjamin 1967; Leibovich 1970) that waves of finite amplitude measured by $\epsilon = O(\kappa^2)$ are possible in a straight tube $Y \equiv 1$. Assuming that such waves have formed a straight section, the effects of varying Y are sought. Hence we shall assume $\epsilon = \kappa^2$ at the outset.

We postpone consideration of viscous effects until § 5. The boundary conditions appropriate to an inviscid flow are

$$\Psi(0, z, t) = 0 \quad \text{and} \quad \Psi(1 + \delta h(x), z, t) = Q, \tag{2}$$

where the constant Q is the volumetric flow rate. We assume that Ψ has a Taylor series expansion about $\delta = 0$, which then permits a transfer of the second boundary condition to the cylinder $y = 1$ by the formula

$$\Psi(1, z, t) + \sum_{n=1}^{\infty} \frac{\delta^n}{n!} h^n(x) \frac{\partial^n \Psi}{\partial y^n}(1, z, t) = Q. \tag{3}$$

Three small parameters have been introduced for inviscid flows. They are α, which represents the slowness of the change in tube area with axial distance; δ, which represents the small total variation of the tube area; and ϵ, which represents the amplitude of the assumed wave motion that is the subject of our investigation.

3. The linearized problem: critical and non-critical flows

If $\epsilon = \delta = 0$, it is seen from the inviscid ($\tilde{\mu} = 0$) form of (1) that a steady cylindrical flow is possible, with arbitrary axial velocity $W(y)$ and arbitrary circulation $\Gamma_s(y)$. We assume that W and Γ_s are given, and represent a flow stable to axisymmetric disturbances, which is anticipated provided that the Howard–Gupta (1962) stability criterion is met. Writing

$$\Psi = \frac{1}{2}\int_0^y W(y)\,dy + \epsilon\psi_0(y,z,t) + \delta\psi_1(y,z)$$

and
$$\Gamma = \Gamma_s(y) + \epsilon\Gamma_0(y,z,t) + \delta\Gamma_1(y,z)$$

then substituting into (1), (2) and (3) we find that

$$\int_0^1 \tfrac{1}{2}W(y)\,dy = Q.$$

If Γ_0 and Γ_1 are then eliminated, the perturbation stream functions are seen to be determined by

$$\left.\begin{aligned} \psi''_{0tt} + 2W\psi''_{0zt} + W^2\psi''_{0zz} - W''[\psi_{0tz} + W\psi_{0zz}] + (\Gamma_s\Gamma'_s/2y^2)\,\psi_{0zz} &= 0,\\ \psi_0(0,z,t) = \psi_0(1,z,t) &= 0, \end{aligned}\right\} \tag{4}$$

where we have used primes to denote differentiation with respect to y, and

$$\left.\begin{aligned} W^2\psi''_{1zz} - WW''\psi_{1zz} + (\Gamma_s\Gamma'_s/2y^2)\,\psi_{1zz} &= 0,\\ \psi_1(0,z) = 0, \quad \psi_1(1,z) &= -\tfrac{1}{2}W(1)\,h(x). \end{aligned}\right\} \tag{5}$$

The problem represented by (4) has a solution of the form

$$\Psi_0 = \phi_0(y)\,A(z,t),$$

where
$$A_t = -c_0 A_z \tag{6}$$

and
$$\left.\begin{aligned} L\phi_0 \equiv \phi''_0 + (W-c_0)^{-2}\,[\Gamma_s\Gamma'_s/2y^2 - (W-c_0)\,W'']\,\phi_0 &= 0,\\ \phi_0(0) = \phi_0(1) &= 0, \end{aligned}\right\} \tag{7}$$

with c_0 a constant. With W and Γ_s specified, c_0 may be regarded as the eigenvalue emerging from problem (7). Of course (6) merely implies that $A = A(z - c_0 t)$. Since the support flow is assumed to be stable, at least two values of c_0, one less than the minimum W and the other greater than the maximum of W, are assured by a theorem of Chandrasekhar (1961, (7, 8*b*)). If W itself does not vanish, then the only possible singular point of (7) occurs at $y = 0$. To assure the existence of regular solutions in (0, 1), it is assumed that $\Gamma_s(0) = 0$, a condition always met by real vortices. Problem (5) has a solution of the form

$$\psi_1 = \theta_1(y)\,h(x),$$

with
$$\alpha^2[M\theta_1]\,h_{xx} \equiv \alpha^2\{\theta''_1 + W^{-2}[\Gamma_s\Gamma'_s/2y^2 - WW'']\,\theta_1\}\,h_{xx} = 0.$$

To $O(\delta)$, the differential equation for ψ_1 is satisfied for arbitrary $M\theta_1$, since the derivatives with respect to z show that the expression above is of order $\delta\alpha^2$. Consideration of higher order terms (specifically, the $O(\delta\alpha)$ contribution to ψ and Γ) shows, however, that the perturbation is not ordered unless $M\theta_1$ vanishes. The problem for θ_1 is therefore

$$M\theta_1 = 0, \quad \theta_1(0) = 0, \quad \theta_1(1) = -\tfrac{1}{2}W(1). \tag{8}$$

If the least eigenvalue c_0 is negative, the flow is said to be subcritical, and if positive it is said to be supercritical (Benjamin 1962). Subcritical flows permit upstream propagation (or $c_0 < 0$) and supercritical flows do not. The dividing case of $c_0 = 0$ is said to be critical. We shall always regard c_0 as the least eigenvalue of (7).

In the case of critical flow, the operators L and M are identical. Since (7) is associated with homogeneous boundary conditions and (8) with inhomogeneous conditions, there cannot be a solution for both ϕ_0 and θ_1 in critical flow. For non-critical flows, this difficulty does not arise, nor does it occur for $\delta = 0$, which is the only case specifically treated in the literature.

Critical flows in other contexts are known to exhibit singular behaviour. Stoker (1957, p. 210), for example, shows that, in the case of water waves on a running stream, the velocity potential has no steady limit and, in fact, it becomes indefinitely large everywhere as $t > \infty$ if the stream is critical. Stoker infers from this that the assumption of small disturbances breaks down for critical flow and that nonlinear effects are required to remove the singular behaviour. A second example is the failure of linearized irrotational gasdynamics as the Mach number approaches unity. The resolution of the singularity at sonic conditions may be resolved by consideration of nonlinear effects.

It appears that the resolution of the present dilemma also requires that nonlinear effects be accounted for. We begin by noting that a solution valid for all c_0 is to be found by adding another term to the expansion previously assumed so that

$$\Psi = \frac{1}{2}\int_0^y W\,dy + \delta^{\frac{1}{2}} f(x)\,\theta_0(y) + \delta h(x)\,\theta_1(y) + \epsilon A(z,t)\,\phi_0(y) + \dots$$

with a similar expansion for Γ.

One then finds that

$$M\theta_0 = 0, \quad \theta_0(0) = \theta_0(1) = 0 \tag{9}$$

and, provided that one chooses $ff_x = \omega_1 h_x$, where ω_1 is a constant to be determined, θ_1 is determined by

$$\left.\begin{aligned} M\theta_1 = \omega_1\{W^{-1}(\theta_0\theta_0''' - \theta_0'\theta_0'') - \tfrac{1}{2}y^{-2}W^{-2}\theta_0^2(2W^{-1}\Gamma_s\Gamma_s')'\},\\ \theta_1(0) = 0, \quad \theta_1(1) = -\tfrac{1}{2}W(1). \end{aligned}\right\} \tag{10}$$

When the flow is not critical only the trivial solution is possible for θ_0, and the problem for θ_1 is as before and has an acceptable solution. For critical flows an eigenfunction solution for θ_0 is possible. With such a θ_0, a second solution to the equation $Mv = 0$ is

$$v = \text{constant} \times \theta_0(y) \int \frac{dy}{\theta_0^2}.$$

The solution for θ_1, apart from an eigenfunction, is

$$\theta_1 = \omega_1 v(y) \int_0^y N(y')\,\theta_0(y')\,dy' + \omega_1\theta_0(y) \int_y^1 N(y')\,v(y')\,dy', \tag{10a}$$

where $$N(y) \equiv W^{-1}(\theta_0\theta_0''' - \theta_0'\theta_0'') - y^{-2}W^{-2}\theta_0^2(\Gamma_s\Gamma_s' W^{-1})'.$$

The boundary condition at $y = 0$ is satisfied and imposition of the condition at $y = 1$ fixes the constant ω_1 to be

$$\omega_1 = -\frac{1}{2}\frac{W(1)}{v(1)\int_0^1 N(y)\,\theta_0(y)\,dy} \tag{11}$$

provided that the denominator is non-zero.

It should be noted that the $\delta^{\frac{1}{2}}$ term added to ψ is unique among the class of fractional powers of δ, in that it is the only term allowing a solution for θ_1.

Thus a solution has been found that holds for all c_0. The extra term added to the original expansion enters only for $c_0 = 0$, and does not appear for $c_0 \neq 0$. The limit of the solution proposed, then, is singular as $c_0 \to 0$. In fact, since the solution θ_1 for $c_0 \neq 0$ is

$$\theta_1(y) = -\tfrac{1}{2}(W(1)/u(1))\,u(y), \tag{12}$$

where $u(y)$ is the solution of $Mu = 0$ that vanishes at $y = 0$, it is clear that the solution (12) breaks down for $u(1) = O(\delta)$. Since $u(1) = O(c_0)$ as $c_0 \to 0$, the solution thus fails for $c_0 = O(\delta)$, and we expect nonlinear effects to come into play in determining the time evolution of the wave, thus forming a smooth transition from $c_0 \neq 0$ to the critical condition.

Before considering nonlinear effects in both near-critical and non-critical flows, it should be noted that, for solid-body rotation and constant axial velocity, $\omega_1 = \infty$. As has been noted earlier (Benjamin 1967; Leibovich 1970) this case is also peculiar in not permitting waves of permanent form. The blocking behaviour at the critical condition found by Chow (1969) for solid-body rotation is probably related to the singular behaviour for ω_1 in this case.

4. Nonlinear effects in inviscid fluids

Our purpose in this section is to obtain equations capable of an asymptotic description of the modifications to the waves of the previous section as ϵ, δ, $\alpha \to 0$ in inviscid fluids. Weak viscous effects are discussed in the following section.

Although the nonlinear and wall interaction perturbations represented by ϵ, α and δ are negligible for limited times, their cumulative effects determine the nature of the disturbance for long times. The multiple-time-scale perturbation methods used are now standard, and the simplest variant (which we employ here) is described by Benney (1966). It is convenient to treat the critical and non-critical case separately.

The dependent variables are formally expanded in the form (suggested by the boundary condition (3) and the considerations of the previous section):

(i) for non-critical flows,

$$\begin{aligned}\psi = \frac{1}{2}\int_0^y W\,dy + \epsilon\phi_0(y)\,A(z,t;\epsilon;\alpha;\delta) + \delta\theta_1 h(x) + \epsilon^2\phi_1 \tfrac{1}{2}A^2 + \epsilon\phi_2 A_{zz} \\ + \epsilon\delta\phi_3 hA + \delta^2\theta_2 h^2(x) + \epsilon\alpha\delta(\phi_4 - \phi_3)\,h_x\int^z A\,dz + O(\epsilon\delta^2) + O(\epsilon^2\delta) \\ + O(\epsilon^3) + O(\delta^3);\end{aligned} \tag{13}$$

(ii) for critical flows,

$$\psi = \frac{1}{2}\int_0^y W\,dy + \epsilon\phi(y)A + \delta^{\frac{1}{2}}\theta_0 f(x) + \epsilon^2[\tfrac{1}{2}\phi_1 A^2 + \phi_2 A_{zz}] + \delta\theta_1 h(x)$$
$$+\epsilon\delta^{\frac{1}{2}}\phi_3 fA + \epsilon\alpha\delta^{\frac{1}{2}}(\phi_4 - \phi_3) f_x \int^z A\,dz + O(\epsilon\delta) + O(\epsilon^2\delta^{\frac{1}{2}}) + O(\epsilon^3), \qquad (14)$$

with similar expansions for the appropriate Γ's (with γ_i replacing ϕ_i, and Ω_i replacing θ_i). Integration by parts is used in deriving the necessary expansions, and the smallness of α is crucial to the separation-of-variables procedure used here. For example, terms like

$$\phi_3 \int hA_z\,dz$$

are encountered, and replaced by

$$\phi_3[hA - \alpha h_x \int A\,dz + O(\alpha^2)],$$

and similarly (from (6))

$$\frac{\partial}{\partial t}[\phi_3 \int hA_z\,dz] = -c_0\phi_3[hA_z - \alpha h_x A + O(\alpha^2)].$$

In (13) and (14) the third-order terms $\epsilon\alpha\delta$ and $\epsilon\alpha\delta^{\frac{1}{2}}$ are retained, but no other third-order terms are considered. This is valid only if $\epsilon \ll \alpha$, and if δ or $\delta^{\frac{1}{2}}$ is small compared with α. We concentrate upon this case, because it yields the simplest expression of geometric effects upon the wave motion. For the form of the complete third-order equations, see Randall (1972). We note that cubic terms, independent of ϵ (e.g. δ^3), result in adjustments of the support flow and do not influence the wave propagation to third order.

With the expansions (13) and (14) there must be associated the equations

$$A_t = -c_0 A_z + \epsilon[c_1 AA_z + c_2 A_{zzz}] + \delta c_3 hA_z + \alpha\delta c_4 Ah_x \qquad (15)$$

and

$$A_t = \epsilon[c_1 AA_z + c_2 A_{zzz}] + \delta^{\frac{1}{2}} c_3 fA_z + \delta^{\frac{1}{2}}\alpha c_4 Af_x, \qquad (16)$$

which permit solutions to be found for the ϕ_i and θ_i. Substitution of (13) and (15) or (14) and (16) into the governing equations (1) produces a sequence of ordinary differential equations for the various ϕ_i and θ_i of the form

$$L\phi_i = F_i(c_i, \phi_j, \theta_k), \quad j,k < i, \qquad (17)$$

and

$$M\theta_i = G_i(\omega_i, \theta_k), \quad k < i.$$

The solution for θ_1 satisfying the appropriate boundary conditions, which are found from (2) and (3), is given by (10*a*) and the corresponding solvability constant ω_1 is given by (11). Higher order θ's may be found in a similar way. Since they are not required for discussion of wave propagation to the order considered, we shall not give them further consideration. The constants c_i, $i \geqslant 1$, appearing in (15), (16) and (17) are given by functionals of ϕ_k and θ_k, $k < i$. These formulae, and equations for determining ϕ_1 and ϕ_2, are given by Leibovich (1970). The functions ϕ_3 and ϕ_4 are new and F_3, F_4, the required boundary conditions and the desired functionals for c_3 and c_4 are given in the appendix.

5. Modifications for weak viscous effects

The modifications required to incorporate weak viscosity in the cylindrical case $\delta = 0$ have been considered elsewhere (Leibovich & Randall 1971). In that treatment, viscous effects were assumed to act only upon the perturbations to the cylindrical support flow. For a truly parallel laminar flow, vanishing of the viscous terms requires that the axial velocity profile be parabolic (in r) and the swirl be a combination of solid-body rotation and a potential vortex. This is not typical of concentrated vortices, and we wish our results to apply to a wider class of support flows. We wish, however, to continue to neglect the viscous terms $8\tilde{\mu}y(yW')''$ and $4\tilde{\mu}y\Gamma''$ that occur in (1), and also to ignore boundary-layer development along the tube walls due to the passage of the wave. The first question is analogous to application of the Orr–Somerfeld equation to not-quite-parallel flows. There are two ways in which it appears that this is a valid procedure in our case (which requires a different justification from that used in stability theory).

(i) When one considers a wave nearly stationary with respect to the wall, it would seem reasonable that small viscous effects due to the wave perturbations will be important as $t \to \infty$, but changes due to the discarded terms are of importance only if the wave propagates over *distances* comparable with $l/\tilde{\mu}$.

(ii) If the support flow is turbulent, the discarded terms could be balanced by turbulent Reynolds stresses, while the dissipative effects upon the wave are assumed to be incorporated in a (constant) eddy viscosity.

The second approach has been used by Miles (1959, p. 571).

Neglect of the boundary layer caused by the wave is discussed by Leibovich & Randall (1971), where it is argued that this is permissible providing that

$$\epsilon \ll (\nu_e/\nu_m)(\nu_m/V_0 l)^{\frac{1}{2}},$$

where ν_e is the eddy viscosity and ν_m the molecular viscosity. In the turbulent case $\tilde{\mu}$ should be based upon ν_e.

Inclusion of the viscous contribution to the equation (either (15) or (16)) for A is of importance only for times of $O(\tilde{\mu}^{-1})$. In the non-critical case, the wave propagates a distance comparable with $c_0 l/\tilde{\mu}$ during such a time interval. Under these circumstances the remark (i) above, which contemplates propagation over distances small compared with $l/\tilde{\mu}$, could not apply. Consequently, we shall not consider dissipation effects in non-critical flows, although they may be of significance in the critical case.

With this preface, we carry over the results of the cylindrical viscous analysis (Leibovich & Randall 1971) only to the case of propagation on critical flows. This requires the addition of the term $\epsilon\tilde{\mu}\phi_5(y)\int A\,dz$ to (14) and the term $\tilde{\mu}c_5 A$ to the right-hand side of (16). The problem for ϕ_5 and the formula determining c_5 are given in the paper cited above. (Replace the subscript 3 in that paper by 5 in order to transform to the notation used here.) We then have equation (15) for non-critical flows and, upon noting (see appendix) that $c_3 = c_4 = c_1$ for critical flows only, we have

$$A_t = \epsilon[c_1 A A_z + c_2 A_{zzz}] + \delta^{\frac{1}{2}} c_1 (fA)_z + c_5 \tilde{\mu} A \qquad (18)$$

for critical flows.

For the remainder of this paper, only non-critical flows will be treated. The critical case is considered in the companion paper.

6. Amplification and decay in non-critical flow

Equation (15) may be put in a more convenient form by transforming to the scaled co-ordinates

$$X = \int^{z} \frac{dz}{c_0 - \delta c_3 h(az)} - t, \quad T = \epsilon t \tag{19}$$

and introducing the functions

$$\mu(X,T) = -\frac{c_4}{\epsilon c_3}\frac{\partial}{\partial X}\log\left(\frac{c_0 - c_3\delta h}{c_0 - c_3\delta h_0}\right) = \frac{c_4}{c_3}\frac{\partial}{\partial T}\log\left(\frac{c_0 - c_3\delta h}{c_0 - c_3\delta h_0}\right), \tag{20a}$$

where $h_0 = h(X_0, T_0)$, X_0 and T_0 being arbitrary reference values, and

$$\lambda = \frac{c_1}{c_0 - c_3\delta h}, \quad \beta = \frac{c_2}{(c_0 - c_3\delta h)^3}. \tag{20b, c}$$

If this is done (15) is replaced by

$$A_T = \lambda A A_X + \beta A_{XXX} + \mu A \tag{21}$$

to the same order of accuracy.

The coefficients λ, β and μ in (21) are slowly varying functions of X and T. Equation (21) is in precisely the form found by Kakutani (1971), his equation (3.10′). An alternative to (21) may be obtained by regarding X and z as independent variables, in which case the form obtained may be identified with Kakutani's equation (3.10).

Consideration of the equation obtained from (19) by replacing λ, β and μ by constants shows that $\mu > 0$ leads to amplification, while $\mu < 0$ leads to damping of the disturbance (Leibovich & Randall 1971). The damped solution for $-\mu$ small is given by Ott & Sudan (1970), and will be reported on by the present authors elsewhere for $-\mu$ not small.

The amplifying case of μ positive and $\mu T \gg 1$ yields a very simple closed-form similarity solution, which is taken up in the next section.

7. Terminal similarity: a nonlinear 'Green's law'

Infinitesimal waves of extreme length are described by linearizing (21) and setting $\beta = 0$. (Finite β accounts for the finite, but long, length scale of the disturbance: although our derivation started by fixing this length scale to be of $O(\epsilon^{\frac{1}{2}})$ in order to retain dispersive effects, we could have rescaled to describe the situation above.) This leads to

$$A_T = \mu A, \tag{22}$$

or

$$A = A_0 \exp\left(\textstyle\int \mu \, dT\right) = A_0 \left(\frac{c_0 - c_3\delta h(X,T)}{c_0 - c_3\delta h_0}\right)^{c_3/c_4}.$$

It will now be shown that, if nonlinear and dispersive effects are retained, (21) predicts a more rapid growth (or decay) than (22). It may be inferred from an obvious modification of the results of Leibovich & Randall (1971) that (if $\lambda \neq 0$, $\beta \neq 0$, but both constant) the momentum and energy of the entire wave train grows or decays like that of the linearized system. That individual waves amplify according to a modified rule seems at first sight surprising, but is explicable by the nonlinear dispersive tendency to form solitons with the consequent concentration of wave momentum and energy.

For shallow-water wave propagation, Kakutani (1971) derives a coefficient

$$\mu = -\tfrac{1}{4}\partial[\log H(X,T)]/\partial T, \tag{23}$$

where H (our notation) is the local depth from the bottom to the mean water line. For this case, A represents the vertical displacement of the free surface from its undisturbed position; from (22) and (23) the result

$$A = A_0(H_0/H)^{\frac{1}{4}} \tag{23a}$$

is obtained for infinitesimal waves. This is known as 'Green's law' (Lamb 1932, § 185).

Numerical solutions to (21) for constant coefficients suggest use of the transformation

$$A = a(X,T)\,G(\eta,T), \tag{24a}$$

where G is to remain of $O(1)$ for large T, and

$$\eta = \left[\frac{a\lambda}{12\beta}\right]^{\frac{1}{2}}\left(X + \tfrac{1}{3}\lambda\int^{T} a\,dT\right), \tag{24b}$$

$$a = a_0 \exp\left(\textstyle\int p\,dT\right). \tag{25}$$

The co-ordinate η is locally the same as the solitary-wave similarity variable of the Korteweg–de Vries equation. It depends upon the slowly varying parameters β and λ of equation (21), and upon the local amplitude $a(X,T)$. The amplitude a, in turn, is assumed to depend upon the function p which is to be determined, and which is assumed to vary slowly with X and T.

We substitute (23)–(25) in (21), and regard the slowly varying functions λ, β, μ and p as constants, to obtain

$$G_{\eta\eta\eta} + 12GG_\eta - 4G_\eta = \frac{1}{\beta}\left[\frac{12\beta}{\lambda a}\right]^{\frac{3}{2}}[G_T + \tfrac{1}{2}p\eta G_\eta + (p-\mu)\,G]. \tag{26}$$

For T large and μ positive, we expect a to be large; if this is true, then an expansion of the form

$$G = \sum_{k=0}^{\infty} a^{-\frac{3}{2}k}\,G_k(\eta)$$

is appropriate. The subsequent details are straightforward and are omitted here. It is found that

$$G = \operatorname{sech}^2\eta + (12\beta)^{\frac{1}{2}}(\lambda a)^{-\frac{3}{2}}\{\tanh\eta + 1 + \operatorname{sech}^2\eta[2\eta - 3 + (\eta^2 + 3\eta - 1) \times \tanh\eta]\} + O(a^{-\frac{5}{2}}) \tag{27}$$

and that the relation

$$p = \tfrac{4}{3}\mu \tag{28}$$

is required to allow G_1 to remain bounded. Thus from (28) and (20)

$$a = a_0\exp\left(\frac{4}{3}\int^{T}\mu\,dT\right) = a_0\left[\frac{c_0 - c_3\,\delta h}{c_0 - c_3\,\delta h_0}\right]^{\frac{4}{3}c_4/c_3}. \tag{29}$$

For the water wave case

$$a = a_0(H_0/H)^{\frac{1}{3}}, \tag{30}$$

which may be contrasted with the linear Green's law (23*a*).

We have carried out computer calculations for equation (21) with $\lambda = 6$, $\beta = 1$ and $\mu = 1$ to assess the validity of the terminal similarity solution. The results of this work, together with a more detailed description of the terminal similarity solution, will be presented elsewhere. We summarize our findings below.

(i) With positive initial conditions, equation (21) produced exactly the same number of solitons as does the Korteweg–de Vries (K dV) equation ($\mu = 0$); initial conditions leading to 1, 2 and 5 solitons were chosen.

(ii) Each emerging wave was well fitted by the Korteweg–de Vries solitary-wave shape.

(iii) For these solutions, the $\frac{4}{3}\mu$ amplification law was rapidly established for *each* soliton: the trajectories of the solitons followed those of KdV solitons with current amplitudes given by (25).

(iv) The amplitude a_0 in (25) was determined by extrapolating to zero time terminal similarity behaviour that emerged from the numerical calculations: the deviation of a_0 from the amplitude of the corresponding Korteweg–de Vries soliton was less than the expected numerical errors.

(v) An almost flat shelf agreeing well in magnitude with that predicted from (27) joined the separating solitons. The last shelf ended in the small-scale oscillations typical of KdV solutions emerging from most initial conditions (Gardner *et al.* 1967; Zabusky 1968). The shelf was strongly reminiscent of the reflected wave reported by Peregrine (1967) and Madsen & Mei (1969).

(vi) Initial conditions with both positive and negative portions may or may not lead to soliton formation depending upon the arrangement of the initial data.

The total increase in amplitude of a given initial distribution consists of the establishment of the amplitudes a_0 for each emerging soliton as well as amplification with rate determined by $\frac{4}{3}\mu$. In general, a_0 is different from the amplitudes of local maxima in the initial data.

We do not attempt a quantitative comparison of the 'nonlinear Green's law' with other solutions of the shallow-water wave equations. The comparison is made difficult since the overall increase in amplitude, as we have noted, depends upon the form of the initial data. Qualitatively, however, we believe the present results agree with others. For example, Madsen & Mei (1969) report exponents in Green's law that differ from the 0·25 linear value; these range from 0·19 to 0·47 in the experiments of Ippen & Kulin (1955) and from 0·15 to 0·30 for their own calculations. The figure 0·33 in our result (30) is not inconsistent with the values quoted, particularly since we expect an overall amplification to fluctuate depending upon the initial conditions.

This work was supported by NASA Grant NGL-33-010-042, monitored by the Lewis Research Center.

Appendix

The equations for ϕ_3 and ϕ_4 for non-critical flows are

$$L\phi_3 = F_3 = c_3 M(y;c_0) + (W-c_0)^{-1}\phi_0\{\theta_1'(P(y;c_0)-P(y;0)) - P'(y;0)\,\theta_1\}$$
$$+\phi_0 y^{-2}(W-c_0)^{-2}\{c_0\Gamma_s\Gamma_s' W^{-1}(W-c_0)^{-1}\theta_1' - \theta_1[W^{-1}\Gamma_s\Gamma_s']'\} \quad \text{(A 1)}$$

and

$$L\phi_4 = F_4 = c_4 M(y;c_0) + (W-c_0)^{-1}\theta_1\{\phi_0'[P(y;0)-P(y;c_0)] - P'(y;c_0)\,\phi_0\}$$
$$-\theta_1 y^{-2}(W-c_0)^{-2}\{c_0\Gamma_s\Gamma_s'\phi_0' W^{-1}(W-c_0)^{-1} + \phi_0[(W-c_0)^{-1}\Gamma_s\Gamma_s']'\}$$
$$+c_0(W-c_0)^{-1}[2P(y;c_0)\,\phi_3 - F_3], \quad \text{(A 2)}$$

where $$M(y;c_0) = [2y^{-2}(W-c_0)^{-3}\Gamma_s\Gamma_s' - (W-c_0)^{-2}W'']\,\phi_0$$

and $$P(y;c_0) = y^{-2}(W-c_0)^{-2}\Gamma_s\Gamma_s' - (W-c_0)^{-1}W''.$$

The boundary conditions are

$$\phi_3(0) = \phi_4(0) = 0$$

and $$\phi_3(1) = \phi_4(1) = -\phi_0'(1).$$

The latter conditions may be replaced by homogeneous ones if we put

$$\tilde{\phi}_j = \phi_j + y\phi_0' \quad (j = 3, 4). \quad \text{(A 3)}$$

In critical flows, one should replace θ_1 by $\theta_0 = \phi_0$, put $c_0 = 0$ in these equations, and replace the above boundary conditions by

$$\phi_3(0) = \phi_4(0) = \phi_3(1) = \phi_4(1) = 0.$$

It is noted that, in critical flows, the problems for ϕ_3 and ϕ_4 are identical to each other and to the problem for ϕ_1.

The constants c_3 and c_4 in non-critical flow are determined by orthogonality conditions

$$\int_0^1 \phi_0 \tilde{F}_j\, dy = 0 \quad (j = 3, 4), \quad \text{(A 4)}$$

required in order that the problems for $\tilde{\phi}_3$ and $\tilde{\phi}_4$ have solutions. Here the $\tilde{F}_j$ are obtained from (A 1) and (A 2) after the substitution of (A 3).

In critical flows, one should replace $\tilde{F}_j$ in (A 4) by F_j. This shows that $c_3 = c_4 = c_1$.

REFERENCES

Benjamin, T. B. 1962 Theory of the vortex breakdown phenomenon. *J. Fluid Mech.* **14**, 593.

Benjamin, T. B. 1967 Some developments in the theory of vortex breakdown. *J. Fluid Mech.* **28**, 65.

Benney, D. J. 1966 Long non-linear waves in fluid flows. *J. Math. & Phys.* **45**, 52.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press.

Chow, C. Y. 1969 Swirling flow in tubes of non-uniform cross-sections. *J. Fluid Mech.* **38**, 843.

Gardner, C. S., Green, J. M., Kruskal, M. D. & Miura, R. M. 1967 Method for solving the Korteweg–de Vries equation. *Phys. Rev. Lett.* **19**, 1095.

HOWARD, L. N. & GUPTA, A. S. 1962 On the hydrodynamic and hydromagnetic stability of swirling flows. *J. Fluid Mech.* **14**, 463.

IPPEN, A. T. & KULIN, G. 1955 *M.I.T. Hydro. Lab. Tech. Rep.* no. 15.

JOHNSON, R. S. 1972 Some numerical solutions of a variable-coefficient Korteweg–de Vries equation (with applications to solitary wave development on a shelf). *J. Fluid Mech.* **54**, 81.

KAKUTANI, T. 1971 Effect of an uneven bottom on gravity waves. *J. Phys. Soc. Japan*, **30**, 272.

KORTEWEG, D. J. & DE VRIES, G. 1895 On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Phil. Mag.* **39** (5), 422–443.

LAMB, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

LEIBOVICH, S. 1970 Weakly non-linear waves in rotating fluids. *J. Fluid Mech.* **42**, 803.

LEIBOVICH, S. & RANDALL, J. D. 1971 Dissipative effects on nonlinear waves in rotating fluids. *Phys. Fluids*, **14**, 2559.

MADSEN, O. S. & MEI, C. C. 1969 The transformation of a solitary wave of finite amplitude over an uneven bottom. *J. Fluid Mech.* **39**, 781.

MILES, J. W. 1959 On the generation of surface waves by shear flows. Part 2. *J. Fluid Mech.* **6**, 568.

OTT, E. & SUDAN, R. N. 1970 Damping of solitary waves. *Phys. Fluids*, **13**, 1432.

PEREGRINE, D. H. 1967 Long waves on a beach. *J. Fluid Mech.* **27**, 815.

RANDALL, J. D. 1972 Two studies of solitary wave propagation in rotating fluids. Ph.D. thesis, Cornell University.

RANDALL, J. D. & LEIBOVICH, S. 1973 The critical state: a trapped wave model of vortex breakdown. *J. Fluid Mech.* **58**, 495.

STOKER, J. J. 1957 *Water Waves*. Interscience.

ZABUSKY, N. J. 1968 Solitons and bound states of the time-independent Schrödinger equation. *Phys. Rev.* **168**, 124.